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# Tracking a Splitting Target in Clutter Using the IMM Methodology

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***Abstract*—We consider the problem of tracking a target that may split, several times, into separate targets, such that each split becomes an autonomous target. This setting arises, for example, when an aircraft launches a series of missiles, or a ballistic missile with multiple warheads breaks into pieces in the reentry phase. The problem is cast into a framework proposed in an earlier work and formulated using a single, generalized state-space model with randomly switching coefficients. Consequently, the states of the original target, as well as of the splits thereof, are tracked efficiently using a single IMM-like algorithm.**

## I. Introduction

Splitting targets are typically encountered in air-to-air missile applications or in situations in which a ballistic missile with multiple warheads breaks into pieces in the reentry phase. Characterized by a sudden increase in the number of targets present in the scene, estimating the states of all targets may be an important task in both military and civilian applications.

A straightforward approach to tracking splitting targets in cluttered environments is to employ standard multiple target tracking methods. These techniques typically combine tracking algorithms that are robust to clutter, such as the Nearest Neighbor (NN) filter [1] or the Probabilistic Data Association (PDA) method [2] with some track initiation logic. Indeed, in [3] a utilization of the Multiple Hypotheses Tracker (MHT) [4] to deal with the case of splitting targets was proposed. Alternatively, the authors of [5] derived a heuristic combination (see [6]) of the Interacting Multiple Model (IMM) filter [7] and the Joint PDA (JPDA) method [8]. In all these cases a single split of the main target was permitted.

Two aspects characterize the problem of tracking a splitting target. First, in practical settings, the number of splits is bounded by some (small) integer. This stems from the physical limitations of any fighter carrying missiles, or any platform equipped with countermeasure flares. Second, the initial conditions of every new target, at the moment of the split, are identical to the current state of the original target.

In this paper we utilize the above assumptions and, following the general approach of [9], formulate the problem of tracking a splitting target using a single state space model with randomly switching coefficients. Such modeling has been shown to allow a unified treatment of various other problems involving uncertain, or intermittent observations [10]–[15], and maneuvering targets [7], [16]–[21].

As is well known, the optimal, in the minimum mean squared error (MMSE) sense, estimator of the state in systems with randomly switching coefficients, or more specifically, in Markov Jump Linear Systems (MJLS), requires exponentially growing resources [17] and is, thus, impractical in most problems of interest. Therefore, suboptimal state estimation algorithms attract special interest of both researchers and practitioners. The IMM filter is perhaps the most famous method that proposes a successful compromise between performance and complexity. Another option is using linear optimal recursive filters [22], [23].

We utilize the formulation of the problem as an MJLS and design an IMM-like algorithm to simultaneously estimate the states of both the original target and the splits thereof. We note that the resulting algorithm does not require any additional logic to deal with track initiation and data association. In addition, the proposed method is not limited to cases of a single split as is demonstrated in the sequel. In this sense, the current paper formalizes and generalizes previously reported approaches.

The remainder of the paper is organized as follows. In Section II we formally define the problem. The standard IMM algorithm is briefly outlined in Section III. The proposed solution is described in Section IV and its performance is demonstrated in Section V. Concluding remarks are provided in Section VI.

## II. Problem Formulation

Consider a target whose state $x_k^0$ evolves in a given surveillance region according to the linear dynamics

$$x_{k+1}^0 = A^0 x_k^0 + C^0 w_k^0, \quad k = 0, 1, \ldots \tag{1}$$

where $\left\{w_k^0\right\}$ is a zero-mean, unit-covariance Gaussian process noise, $x_0^0$ is a Gaussian random vector with mean $\bar{x}_0$ and covariance $P_0$, and $A^0$ and $C^0$ are deterministic matrices representing the state dynamics and process noise covariance, respectively. At any sampling time $k$, the target may split, with probability $p$, into two targets, or continue evolving according to the original dynamics without splitting with probability $1-p$. In the former case, the original target continues following the dynamics (1) and the new target begins evolving according to the independent dynamics

$$x_{k+1}^i = A^i x_k^i + C^i w_k^i, \quad k = k_i, k_i+1, \dots \tag{2}$$

Here, $x_k^i$ denotes the state of the new target after the $i$-th split, $\{w_k^i\}$ is the corresponding Gaussian zero-mean, unit-covariance process noise and $x_{k_i}^i = x_{k_i}^0$ (with probability 1) where $k_i$ denotes the time of the of $i$-th split.

We assume that the total number of splits is upper-bounded by a known constant $L_{\max}$ such that after the $L_{\max}$-th split there are $L_{\max}+1$ targets $x_k^0, x_k^1, \dots, x_k^{L_{\max}}$ driven by $w_k^0, w_k^1, \dots, w_k^{L_{\max}}$, respectively. The process noise sequences of the different targets are taken to be mutually independent.

Let $N_k$ denote the number of targets at time $k$. It follows from the previous discussion that $\{N_k\}$ is a Markov chain on $\{1, \dots, L_{\max}+1\}$ with transition probability matrix (TPM)

$$\begin{pmatrix} 1-p & p & 0 & \cdots & 0 \\ 0 & 1-p & p & 0 & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & 0 & 1 \end{pmatrix} \tag{3}$$

and initial condition $\mathbb{P}\{N_0 = 1\} = 1$.

Each independently evolving target is measured through the following linear Gaussian channel

$$y_k^i = H^i x_k^i + G^i v_k^i, \quad i = 0, \dots, L_{\max}, \tag{4}$$

where $\{v_k^i\}$ is the corresponding Gaussian zero-mean, unit-covariance measurement noise. The measurement noise sequences corresponding to the different measurement channels are assumed to be mutually independent and independent of all previously defined random quantities. It is also assumed that the measurements are unlabeled, such that it is not known a-priori which measurement corresponds to which target.

At each time, in addition to the target measurements, $\{y_k^i,\, i = 1, \dots, L_{\max}\}$, a number of clutter measurements are obtained. These will be denoted as $y_{k,\mathrm{cl}}^j$, $j = 1, \dots, M_k - N_k$, where $M_k$ is the total number of measurements at time $k$. Clutter measurements originate from false targets and do not carry any information about the target of interest. They are, however, indistinguishable from true detections. At each time, the clutter measurements are assumed to be independent of each other, of the clutter measurements at other times, and of the true states and observations. In addition, we assume that these measurements are uniformly distributed in the surveillance region, which is a common assumption in such applications [2].

For simplicity, we assume that the measurement channels are perfect, in the sense that there are no missed observations. This assumption may be relaxed, as explained in the sequel.

The goal of this paper is to devise an efficient sequential algorithm for estimating the states of the original target and of the splits thereof using all the available measurements. Since the MMSE-optimal estimator requires resources that grow exponentially in time, we, inevitably, consider suboptimal approaches. In the sequel we show how the defined problem may be cast into a single state space formulation with randomly switching coefficients and, consequently, solved using a single IMM-like algorithm yielding the estimates of all the targets.

## III. Background on IMM

The IMM method estimates the state of the following system:

$$x_{k+1} = A(\theta_k) x_k + C(\theta_k) w_k \tag{5}$$
$$y_k = H(\theta_k) x_k + G(\theta_k) v_k. \tag{6}$$

Here, $\{w_k\}$ and $\{v_k\}$ are independent, white, zero-mean, unit-covariance Gaussian sequences, $x_0$ is a Gaussian random vector with known mean and covariance matrix, and $\{\theta_k\}$ is a Markov chain on $\{1, \dots, r\}$ with some known TPM, $(p_{ij})$, and initial distribution vector.

At time $k$, the algorithm recursively estimates $x_k$ using $y_0, \dots, y_k$ providing an approximation of the MMSE solution. The main idea underlying the IMM algorithm is to maintain a bank of primitive Kalman filters, each matched to a different model in the given model set (different value of $\theta_k$). At step $k$, the $j$-th filter produces a local estimate $\hat{x}_{k,j}$ with an associated error covariance $P_{k,j}$ using its initial estimate $\hat{x}_{k-1,j}^{\mathrm{init}}$ and the associated covariance $P_{k-1,j}^{\mathrm{init}}$, which are generated externally, and the current measurement $y_k$, which gets processed by all KFs in the bank. In addition, each filter produces a current value of its own (model-matched) likelihood function $\Lambda_{k,j}$. The key element of the IMM scheme is the interaction block that generates, using all local estimates, covariances, and likelihoods from the previous cycle, individual initial conditions for each of the primitive filters in the bank.

The steps of the algorithm are summarized as follows.

### A. Mixing Probabilities

For $i, j = 1, \dots, r$ compute

$$\begin{aligned} \mu_{k-1}(i \mid j) &\triangleq \mathbb{P}\{\theta_{k-1} = i \mid \theta_k = j, \mathcal{Y}_{k-1}\} \\ &= \frac{1}{c_j} p_{ij} \mu_{k-1}(i), \end{aligned} \tag{7}$$

where $c_j$ is a normalizing constant, $\mu_k(i) \triangleq \mathbb{P}\{\theta_k = i \mid \mathcal{Y}_k\}$ is computed according to (10) below, and $\mathcal{Y}_k \triangleq \{y_0, \dots, y_k\}$.

### B. Mixing Step

For $j = 1, \dots, r$ compute the initial state estimate for the filter matched to $\theta_k = j$

$$\hat{x}_{k-1,j}^{\mathrm{init}} = \sum_{i=1}^{r} \hat{x}_{k-1,i} \mu_{k-1}(i \mid j) \tag{8}$$

and the corresponding covariances.

### C. Mode-Matched Filtering

For $j = 1, \ldots, r$, using (8) and the corresponding covariance, compute the mode-matched estimate $\hat{x}_{k,j}$ and $P_{k,j}$ as well as the likelihood $\Lambda_{k,j}$, which is approximated as Gaussian

$$\Lambda_{k,j} = \mathcal{N}(y_k; \hat{y}_{k,j}, S_{k,j}), \tag{9}$$

where $\hat{y}_{k,j}$ and $S_{k,j}$ are the predicted measurement and innovation covariance computed by the $j$-th filter using the initial conditions (8).

### D. Mode Probability Update

Compute

$$\mu_k(j) = \frac{1}{c}\Lambda_{k,j}\sum_{i=1}^{r} p_{ij}\mu_{k-1}(i), \quad j = 1, \ldots, r \tag{10}$$

where $c$ is a normalization factor.

### E. Output Computation

At time $k$, the algorithm's output is obtained as a fused version of the local estimates:

$$\hat{x}_k = \sum_{j=1}^{r} \hat{x}_{k,j}\mu_k(j). \tag{11}$$

The associated covariance is computed in a similar manner.

## IV. The Proposed Solution

In this section, following the rationale proposed in [9], we show how to efficiently solve the problem of tracking a splitting target using a single IMM-like algorithm. To this end, we need to define the Markov mode sequence $\{\theta_k\}$ and specify the matrices $A(\theta_k)$, $C(\theta_k)$, $H(\theta_k)$, and $G(\theta_k)$.

### A. Mode Set and Evolution

As mentioned above, at each time instant $k$, we collect $M_k$ detections of which only $N_k$ correspond to true targets and the rest to clutter. Consequently, for each possible value of $N_k$, there are $M_k!/(M_k - N_k)!$ different hypotheses corresponding to the identities of the $N_k$ true targets among the $M_k$ measurements. Since the number of targets $N_k$ is known to be between 1 and $L_{\max} + 1$, the total number of hypotheses at time $k$ is given by

$$r_k = \sum_{i=1}^{L_{\max}+1} \frac{M_k!}{(M_k - i)!}. \tag{12}$$

We therefore define a Markov chain $\{\theta_k\}$ on $\{1, \ldots, r_k\}$, such that states $1, \ldots, M_k$ correspond to the case in which the target has not yet split, states $M_k + 1, \ldots, M_k + M_k!/2!$ correspond to the case in which the target has split once, and so on.

Although $r_k$ is a large number even for humble values of $M_k$, it should be noted that combinatorial number of hypotheses is standard in problems involving multi-target scenarios in cluttered environment and cannot be avoided by using standard methods based on, e.g., JPDA.

The TPM of the resulting mode sequence to be used in the IMM is designed using the assumption that, a-priori, for a given number of measurements $M_k$, any ordering of the true detections and clutter measurements is equiprobable. Hence, the TPM is obtained by multiplying each entry of (3) by an all-ones matrix of appropriate dimensions. For example, the TPM for the case of $L_{\max} = 2$ and $M_k = 4$ is given by

$$\begin{pmatrix} \frac{1-p}{4}\mathbf{1}_{4\times4} & \frac{p}{12}\mathbf{1}_{4\times12} & \mathbf{0}_{4\times24} \\ \mathbf{0}_{12\times4} & \frac{1-p}{12}\mathbf{1}_{12\times12} & \frac{p}{24}\mathbf{1}_{12\times24} \\ \mathbf{0}_{24\times4} & \mathbf{0}_{24\times12} & \frac{p}{24}\mathbf{1}_{4\times24} \end{pmatrix}, \tag{13}$$

where $\mathbf{1}_{m\times n}$ and $\mathbf{0}_{m\times n}$ correspond to $m \times n$ matrices of all ones and zeros, respectively. Here, we utilize the fact that the number of modes corresponding to 0, 1, and 2 splits is 4, 12 and 24, respectively (see the discussion before (12)). Generalization to arbitrary values of $L_{\max}$ and $M_k$ is straightforward.

### B. Augmented System Dynamics

To specify a single state evolution equation we define an augmented state as a columnwise concatenation of the individual states $x_k^0, x_k^1, \ldots, x_k^{L_{\max}}$. Likewise, the augmented process noise is obtained by concatenating $w_k^0, w_k^1, \ldots, w_k^{L_{\max}}$ into a single process noise vector. It remains to describe the structure of the matrices $A(\theta_k)$ and $C(\theta_k)$.

Before the first split, there is a single target following the nominal dynamics (1). At this stage, the remaining $L_{\max}$ targets may be viewed as following the same dynamics, driven by the same process noise and initialized with the same initial conditions. Thus, the matrices corresponding to $\theta_k \in \{1, \ldots, M_k\}$ are given by

$$A(\theta_k) = \begin{pmatrix} A^0 & 0 & \cdots & 0 \\ A^0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^0 & 0 & \cdots & 0 \end{pmatrix} \tag{14}$$

and similarly for $C(\theta_k)$. After the first split, there are 2 targets evolving in an autonomous manner and $L_{\max} - 1$ targets following the dynamics equation of the original target. Therefore, the matrices corresponding to $\theta_k \in \{M_k + 1, \ldots, M_k + M_k!/2!\}$ are given by

$$A(\theta_k) = \begin{pmatrix} A^0 & 0 & \cdots & 0 \\ 0 & A^1 & \cdots & 0 \\ A^0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^0 & 0 & \cdots & 0 \end{pmatrix} \tag{15}$$

and similarly for $C(\theta_k)$. Matrices corresponding to additional splits are defined in a similar manner.

### C. Augmented Measurement Equation

Similarly to the examples in [9], [23], [24], the matrices $H(\theta_k)$ and $G(\theta_k)$ are affected by $\theta_k$ due to the data association ambiguity that stems from the multiple targets evolving in cluttered environment. We concatenate all the measurements acquired at time $k$ to obtain the augmented measurement and define similarly the augmented measurement noise.

As it is not known a-priori which measurements are the true target detections, their locations in the augmented measurement vector are unknown. We thus obtain the following values for $H(\theta_k)$ and $G(\theta_k)$ for $\theta_k \in \{1, \ldots, M_k\}$

$$\{H(\theta_k), G(\theta_k)\} = \begin{cases} \left\{ \mathrm{diag}\begin{pmatrix} H^0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \mathrm{diag}\begin{pmatrix} G^0 \\ G_{\mathrm{cl}} \\ \vdots \\ G_{\mathrm{cl}} \end{pmatrix} \right\}, & \theta_k = 1 \\ \qquad\vdots \qquad\qquad \vdots & \\ \left\{ \mathrm{diag}\begin{pmatrix} 0 \\ \vdots \\ 0 \\ H^0 \end{pmatrix}, \mathrm{diag}\begin{pmatrix} G_{\mathrm{cl}} \\ \vdots \\ G_{\mathrm{cl}} \\ G^0 \end{pmatrix} \right\}, & \theta_k = M_k. \end{cases} \tag{16}$$

Here, $G_{\mathrm{cl}}$ is the square-root of the covariance matrix associated with the uniformly distributed clutter.

Consider, e.g., the first realization of $\theta_k$ in (16). It corresponds to the case where the first of the $M_k$ acquired measurements is the true target measurement, $y_k^0$, generated according to (4). All other $M_k - 1$ measurements are clutter. That exactly one of the $M_k$ observations is target-originated is reflected in (16) by the fact that exactly one of the blocks of $H(\theta_k)$ is set to $H^0$, with all others being set to 0. Note that we have assumed that the true measurement is always present in the acquired measurement set. To extend the treatment to the case where true measurement may be missing, we need to augment the previous set of values of the matrices $H(\theta_k)$ and $G(\theta_k)$ with the option $\{\mathbf{0}, I_{M_k} \otimes G_{\mathrm{cl}}\}$, where $\otimes$ denotes the Kronecker product. For simplicity, hereafter we assume that there are no missing observations.

Similarly, the feasible values of $H(\theta_k)$ and $G(\theta_k)$ corresponding to $\theta_k \in \{M_k + 1, \ldots, M_k + M_k!/2!\}$ are

$$\{H(\theta_k), G(\theta_k)\} = \begin{cases} \left\{ \mathrm{diag}\begin{pmatrix} H^0 \\ H^1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \mathrm{diag}\begin{pmatrix} G^0 \\ G^1 \\ G_{\mathrm{cl}} \\ \vdots \\ G_{\mathrm{cl}} \end{pmatrix} \right\}, & \theta_k = M_k + 1 \\ \qquad\vdots \qquad\qquad \vdots & \\ \left\{ \mathrm{diag}\begin{pmatrix} 0 \\ \vdots \\ 0 \\ H^0 \\ H^1 \end{pmatrix}, \mathrm{diag}\begin{pmatrix} G_{\mathrm{cl}} \\ \vdots \\ G_{\mathrm{cl}} \\ G^0 \\ G^1 \end{pmatrix} \right\}, & \theta_k = M_k + M_k!/2!. \end{cases} \tag{17}$$

Matrices corresponding to additional values of $\theta_k$ are defined in a similar manner.

### D. Designing the IMM

To implement a standard IMM for the described node sequence, we maintain a bank of KFs each matched to a different combination of the valid values of $A(\theta_k)$ and $C(\theta_k)$ with those of $H(\theta_k)$ and $G(\theta_k)$. The transitions between different modes are captured by a TPM of the form of (3) and the mode sequence is initialized by assigning equal probability to all the modes corresponding to the "no split" case and zero to the remaining ones.

Note that unless $M_k$ is constant, the number of feasible modes and, consequently, the number of primitive KFs and the dimensions of the TPM at each time change. Therefore, the IMM mode set and the TPM should be recalculated upon receiving every measurement set.

We mention in passing that in the present case direct utilization of the IMM scheme may only be made in an approximate manner. This is since the mode-matched likelihoods of the standard IMM routine are approximated as Gaussian as opposed to the standard assumption of uniformly distributed clutter. One may modify the standard IMM by computing the correct likelihoods, or, alternatively, use the standard algorithm despite the above mismatch and obtain an IMM-*like* solution. We pursue the second option in the next section.

## V. Numerical Example

To demonstrate the proposed algorithm we simulate a target such that the state in (1) comprises the target's position and velocity, $x_k = [p_k \ v_k]^T$, and follows the discrete white noise acceleration (DWNA) model [25], specified by

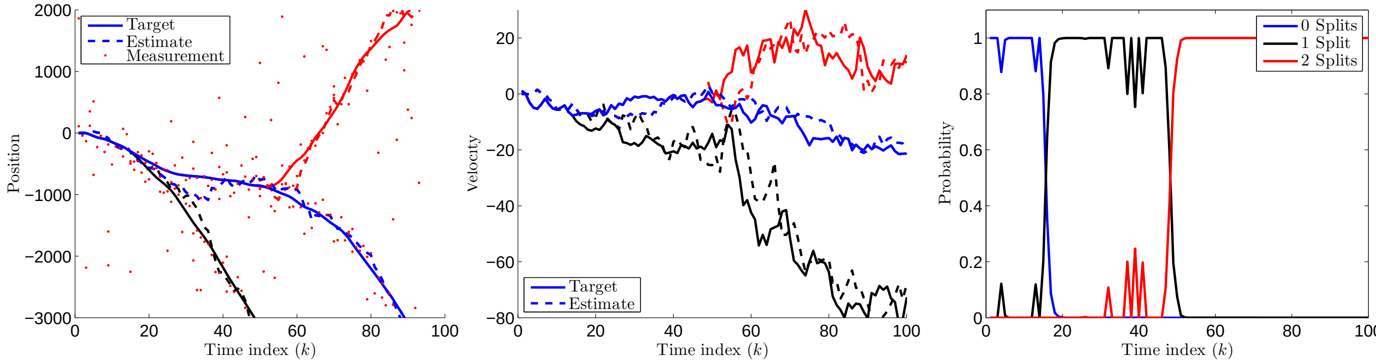
$$A^0 = \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix}, \; B^0 = \begin{pmatrix} T^2/2 \\ T \end{pmatrix} \sigma_0, \tag{18}$$

where $\sigma_0$ is some nominal process noise intensity.

The split targets follow the same dynamics up to the values of the process noise. Noisy position-only detections of each of the autonomous targets are available namely, $H^i = [1 \quad 0]$ and $G^i = \sigma_v$ for all $i = 0, \ldots, L_{\max}$.

In the examples below the following common parameters were used: $L_{\max} = 2$, $p = 0.05$, $\sigma_0 = 0.3$ m/s$^2$, $\sigma_1 = \sigma_2 = 1$ m/s$^2$, $\sigma_v = 200$ m, $T = 5$ s. Two to five clutter measurements were uniformly generated about the origin with standard deviation of $30\sigma_v$. In addition, the initial state is $x_0 = [0 \quad 0]^T$, and $P_0$ is an all-zero matrix. In order to concentrate on the tracking performance and not on initialization capabilities of the algorithm, the filter is initialized in a perfect manner.

The positions and velocities of the targets, accompanied by the corresponding estimates, are presented in Figs. 1a–1b. The probabilities of the number of splits, calculated by the algorithm, are presented in Fig. 1c. It is readily seen that the algorithm is capable of successfully estimating all targets.




**(a) True and estimated positions.** **(b) True and estimated velocities.** **(c) Split probabilities.**

Fig. 1: The proposed algorithm's performance.

## VI. Conclusion

We presented an algorithm for tracking a target that may split, several times, into separate targets. Using the ideas presented in a previous work, we showed how to cast the problem into a unified framework of generalized state-space model with randomly switching coefficients. Consequently, the states of the original target, as well as of the splits thereof, were tracked efficiently using a single IMM-like algorithm defined over an augmented mode-set. To the best of the authors' knowledge, no algorithm that is capable of dealing with multiple splits is available in the literature, and existing solutions for less general problems are rather heuristic. In addition, the idea presented herein may be generalized to problems where several splits occur at the same time as well as to cases where the target splits themselves may also split.

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